LECTURE 3

Minimality and ergodicity

Given a dynamical system, topological or measure-preserving, in this lecture we study proper regions of the state space which stay invariant under the dynamics. More precisely, we are interested in the absence of such regions, since their existence would contradict the “ergodic hypothesis” as formulated by Boltzmann. Indeed, in this case the phase space motion is confined to that region hence cannot reach every other state. Also “time mean equals space mean” would fail since the phase space motion does not spend any time outside of this invariant region.

1. Invariant sets for topological systems

We begin with some definitions for a topological system \((K, T)\). For a point \(x \in K\) and a subset \(A \subseteq \mathbb{N}_0\) (or \(A \subseteq \mathbb{Z}\) if \(T\) is invertible) we define

\[ T^A x := \{ T^n x : n \in A \}, \]

and call the set \(\text{orb}_+(x) := T^\mathbb{N}_0 x\) the orbit of \(x\). A subset \(F \subseteq K\) is called (forward) invariant if \(\text{orb}_+(x) \subseteq F\) for each \(x \in F\). If \(T\) is invertible, then the two-sided orbit of \(x\) is \(\text{orb}(x) := T^\mathbb{Z} x\), and a subset \(F \subseteq K\) is two-sided invariant if \(\text{orb}(x) \subseteq F\) for each \(x \in F\). A closed, invariant set \(F\) yields, by restriction to \(F\), a subsystem \((F, T|_F)\), denoted by \((F, T)\). In the same manner two-sided invariant sets of an invertible system yield invertible subsystems. If the system \((K, T)\) is invertible, then any non-empty, closed, invariant set contains a closed, non-empty, two-sided invariant set (see Exercise 3.1). A point \(x \in K\) is called topologically transitive if \(\text{orb}_+(x) = K\), and topologically two-sided transitive if \(\text{orb}(x) = K\). Finally, the system \((K, T)\) is called minimal if the only closed, invariant sets are the trivial ones, \(\emptyset\) and \(K\).

Remark 3.1. The closure of an invariant set is easily seen to be invariant, so in particular \(\overline{\text{orb}_+}(x)\) is an invariant set, and the system \((K, T)\) is minimal if and only if each of the points \(x \in K\) is topologically transitive. Analogously, an invertible system \((K, T)\) is minimal if and only if each of the points \(x \in K\) is topologically two-sided transitive (see Exercise 3.2). This gives an informal relation to the quasi ergodic hypothesis of the Ehrenfests since in such systems the orbit of every point comes arbitrarily close to any other point.

We introduce the following terminology. Let \((K, T)\) and \((L, S)\) be topological systems. A homomorphism between the \((K, T)\) and \((L, S)\) is a continuous map \(\pi : K \rightarrow L\) such that \(\pi \circ T = S \circ \pi\). We denote this by writing \(\pi : (K, T) \rightarrow (L, S)\). Such a homomorphism is called an automorphism if it is also a homeomorphism between \(K\) and \(L\). Automorphisms take orbits to orbits and closed, invariant sets to closed, invariant sets. This can be exploited in the following characterization of minimality for group rotations.
**Proposition 3.2.** Let $G$ be a compact group and let $a \in G$. The following assertions are equivalent.

(i) The group rotation system $(G, a)$ is minimal.

(ii) The subsemigroup $\{a^n : n \in \mathbb{N}_0\}$ is dense in $G$.

(iii) The cyclic subgroup $\langle a \rangle = \{a^n : n \in \mathbb{Z}\}$ is dense in $G$.

Minimality, therefore, is a strong condition for such systems. For example, it implies that the group $G$ is abelian, see Exercise 3.3.

**Proof.** The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) are clear. To prove that (iii) implies (i) let $x \in G$. Then $\text{orb}(x) = \rho_x \text{orb}(1) = \rho_x \langle a \rangle$, where $\rho_x$ is the multiplication by $x$ from the right and hence an automorphism of $(G, a)$. This implies $\overline{\text{orb}(x)} = G$, i.e., every point in $G$ is topologically two-sided transitive. By Remark 3.1, the proof is complete.

Important examples of minimal systems are provided by certain torus rotations.

**Proposition 3.3** (Kronecker). The system $(\mathbb{T}, a)$ is minimal if and only if $a \in \mathbb{T}$ is not a root of unity, i.e., if the cyclic subgroup $\langle a \rangle$ is infinite.

**Proof.** If $a$ is a root of unity, then $\text{orb}_+(1)$ is finite and invariant, so that $(\mathbb{T}, a)$ cannot be minimal. If $a$ is not a root of unity, then by Exercise 2.2(a) $\text{orb}_+(1)$ is dense in $\mathbb{T}$, and by Proposition 3.2 we obtain the minimality of $(\mathbb{T}, a)$.

The following result is a useful characterization of minimality using open sets instead of points.

**Proposition 3.4** (Characterization of minimality). A topological system $(K, T)$ is minimal if and only if for each non-empty, open set $U \subset K$ there is $N \in \mathbb{N}_0$ such that

$$K = \bigcup_{n=0}^{N} T^{-n} U.$$  

(3.1)

**Proof.** Suppose $(K, T)$ is minimal, and let $U \subset K$ be non-empty and open. Consider the set

$$F := \bigcap_{n \in \mathbb{N}_0} T^{-n} (K \setminus U)$$

of points which never visit $U$. This is a closed, invariant set and $F \neq K$ since $U \cap F = \emptyset$. It follows that $F = \emptyset$, i.e., $K = \bigcup_{n \in \mathbb{N}_0} T^{-n} U$, and by compactness we can take a finite subcover. This proves (3.1).

Conversely, let $F$ be a non-trivial, closed, invariant subset of $K$, then $U := K \setminus F$ is non-empty, open and violates condition (3.1).

It turns out that in a minimal system every point visits each non-empty, open set even infinitely often. Moreover, the sequence (or set) of visiting times has some structure. This is the content of the next proposition.

**Proposition 3.5.** Let $(K, T)$ be minimal, and let $U \subset K$ be a non-empty, open set. Then for each $x \in K$ the set

$$R_U(x) := \{n \in \mathbb{N} : T^n x \in U\}$$
of return times* to $U$ is syndetic (or relatively dense), i.e., there is $N \in \mathbb{N}$ such that $[k,k+N] \cap R_U(x) \neq \emptyset$ for every $k \in \mathbb{N}$.

Proof. By Proposition 3.4 there is $N \in \mathbb{N}$ such that $K = \bigcup_{j=0}^{N} T^{-j}U$. This implies that for each $x \in K$ and $k \in \mathbb{N}$ there is $j \in \{0, \ldots, N\}$ such that $T^kx \in T^{-j}U$, meaning $x \in T^{-(k+j)}U$. It follows that $R_U(x) \cap [k, k+N] \neq \emptyset$.

A point $x \in K$ is called almost periodic if for each non-empty, open set $U \subset K$ the set of return times $R_U(x)$ is syndetic. By the above, in a minimal system every point is almost periodic.

Proposition 3.6 (Characterization of almost periodic points). For a topological system $(K,T)$ and a point $x \in K$ the following assertions are equivalent.

(i) The point $x$ is almost periodic.
(ii) The set $\text{orb}_+(x)$ is minimal.
(iii) The point $x$ is contained in a minimal subsystem.

Proof. The implication (ii)⇒(iii) is trivial and (iii)⇒(i) follows from Proposition 3.5, so we only need to prove (i)⇒(ii). Let $U$ be an open set containing $x$, and let $F := \text{orb}_+(x)$. Since $x$ is almost periodic, there is $N \in \mathbb{N}$ such that $T^kx \in \bigcup_{j=0}^{N} T^{-j}U \subset \bigcup_{j=0}^{N} T^{-j}U$ for each $k \in \mathbb{N}$. It follows that $F \subset \bigcup_{j=0}^{N} T^{-j}U$. As a consequence, for any $y \in F$ there is $j \in \{0, \ldots, N\}$ with $T^jy \in U$. This, being true for every open neighbourhood $U$ of $x$, implies $x \in \text{orb}_+(y)$ and $\text{orb}_+(x) = \text{orb}_+(y)$. Minimality of $\text{orb}_+(x)$ follows.

Here is a strong form of recurrence for points in minimal systems.

Proposition 3.7 (Almost periodic points in metrizable systems are recurrent). Let $(K,T)$ be a topological system with metrizable $K$, and let $x \in K$ be almost periodic. Then $x$ is recurrent, i.e., there is a subsequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ such that $T^{n_k}x \to x$ as $k \to \infty$.

We leave the proof as Exercise 3.4.

Proposition 3.8. Let $(K,T)$ be a topological system.

(a) Let $(L,S)$ be another topological system and let $\pi: (K,T) \to (L,S)$ be a homomorphism. If $x \in K$ is almost periodic in $(K,T)$, then so is $\pi(x)$ in $(L,S)$.
(b) Suppose $(L,T)$ is a subsystem of $(K,T)$. If $x \in K$ is almost periodic in $(K,T)$, then it is almost periodic in $(L,T)$.

We leave the proof of this result again as Exercise 3.4.

Corollary 3.9. In a compact group rotation system $(G,a)$ every point is almost periodic.

Proof. The subgroup $H := \langle a \rangle$ provides the group rotation subsystem $(H,a)$, which is minimal by Proposition 3.2. This implies that 1 is almost periodic, thus for each $x \in K$ the homomorphic image $x = \rho_x(1)$ is almost periodic.

The following result is known as Birkhoff’s Recurrence Theorem[1].

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*Of course, if $x \notin U$ this should be called the set of visiting times, but the terminology return times is more widespread.

Theorem 3.10 (Birkhoff). Each topological system \((K, T)\) contains a minimal subsystem and therefore an almost periodic point. In particular, if \(K\) is metrizable, then it contains at least one recurrent point.

Proof. Consider the set \(F := \{F : F \subset K \text{ closed, invariant, } F \neq \emptyset\}\), which is partially ordered by inclusion \(A \subset B\). This partially ordered sets satisfies the conditions of Zorn’s lemma, and therefore has a minimal element \(F\), which provides a minimal subsystem \((F, T)\). By Proposition 3.6, any point \(x \in F\) is almost periodic. The rest follows from Proposition 3.7. ■

2. Invariant sets for measure-preserving systems

Because of the presence of null sets (which are anyway negligible), the whole business of invariant sets for measure-preserving systems becomes somewhat more subtle. We begin with some terminology.

Let \((X, \mu)\) be a probability space and \(A, B \subset X\) measurable. We say that \(A \subset B\) up to a null set if \(\mu(B \cap A) = 0\). Analogously, we say that \(A = B\) up to a null set if \(\mu(A \Delta B) = 0\), where \(A \Delta B = (A \setminus B) \cup (B \setminus A)\) is the symmetric difference of \(A\) and \(B\). The simple inequality

\[
|\mu(A) - \mu(B)| \leq \mu(A \Delta B)
\]

will be very useful. Note that being equal up to a null set defines an equivalence relation on measurable sets.

Definition 3.11 (Invariant sets). Let \((X, \mu, T)\) be a measure-preserving system. A measurable set \(A \subset X\) is called (\(T\)-)invariant if \(A = T^{-1}A\) up to a null set.

Remark 3.12. By the measure-preserving property, a measurable set \(A\) is invariant if and only if \(T^{-1}(A) \subset A\) holds up to a null set, and this holds if and only if \(A \subset T^{-1}(A)\) holds up to a null set, see Exercise 3.7

The set of invariant sets is denoted by \(\mathcal{X}_T\), and it is a sub-\(\sigma\)-algebra of the underlying \(\sigma\)-algebra \(\mathcal{X}\) (see Exercise 3.7), called the invariant sub-\(\sigma\)-algebra. In particular, we have \(A^c \in \mathcal{X}_T\) if \(A \in \mathcal{X}_T\). A non-trivial invariant set \(A\) thus gives rise to two systems \((X, \mu_A/\mu(A), T)\) and \((X, \mu_{A^c}/\mu(A^c), T)\), where for \(B \in \mathcal{X}\) the measure \(\mu_B\) is defined by

\[
\mu_B(C) := \mu(B \cap C) \quad \text{for each } C \in \mathcal{X}.
\]

Definition 3.13 (Ergodicity). The measure-preserving system \((X, \mu, T)\), and also the transformation \(T\), are called ergodic if \(\mu(A) \in \{0, 1\}\) holds for each invariant set \(A \subset X\).

In Lecture 1 we defined ergodicity via a fixed space condition. In Proposition 3.21 below we will show that this is equivalent to the previous definition.

We now present the first examples of ergodic (and non-ergodic) systems.

Example 3.14 (Finite systems). Let \(X\) be a finite set with the normalized counting measure \(\mu\) and let \(T : X \to X\) be bijective. Then \(T\) is ergodic if and only if it is a cyclic permutation.

Example 3.15 (Identity is (almost) never ergodic). Let \((X, \mu)\) be a probability space. Then \(\text{id} : X \to X\) is ergodic if and only if there is no measurable set \(A \subset X\) with \(0 < \mu(A) < 1\). This shows that, in contrary to the topological case where
every system contains a minimal subsystem, not every measure-theoretic system has an ergodic subsystem. For example, \(([0,1], \lambda, \text{id})\) has no ergodic subsystem.

An important class of ergodic transformations are (Bernoulli) shifts (cf. Example 2.14). We need the following auxiliary result, for a proof see [BPM, Thm. 11.4 and Cor.].

**Lemma 3.16.** Let \((X, \mathcal{X}, \mu)\) be a probability space and let \(\mathcal{E}\) be an algebra of sets and a generator of the \(\sigma\)-algebra \(\mathcal{X}\). Then for every set \(A \in \mathcal{X}\) and for every \(\varepsilon > 0\) there is \(B \in \mathcal{E}\) such that \(\mu(A \triangle B) < \varepsilon\).

**Proposition 3.17** (Shifts are ergodic). Let \((Y, \nu)\) be a probability space. Both the one-sided and the two-sided shift with state space \((Y, \nu)\) are ergodic.

**Proof.** We consider only the case of one-sided shifts \((X, \mu, T)\), the invertible case being analogous. Let \(\mathcal{E}\) be the algebra of finite unions of cylinder sets (which generates the product \(\sigma\)-algebra \(\mathcal{X}\)). Let \(A \in \mathcal{X}\) be an invariant set, let \(\varepsilon > 0\) be arbitrary, and let \(B \in \mathcal{E}\) be such that \(\mu(A \triangle B) < \varepsilon\) (use Lemma 3.16). Then \(B\) is of the form

\[
B = B_N \times Y \times Y \cdots, \quad \text{where } B_N \subset Y^N.
\]

Note that \(T^{-n}B = Y^n \times B_N \times Y \times Y \cdots\), so for every \(n > N\) we have

\[
\mu(B \cap T^{-n}B) = \mu(B) \cdot \mu(T^{-n}B) = \mu(B)^2
\]

by the definition of the product measure. Thus we obtain by (3.2) for every \(n > N\)

\[
|\mu(A) - \mu(A)|^2 \leq |\mu(A) - \mu(B \cap T^{-n}B)| + |\mu(B \cap T^{-n}B) - \mu(A)|^2
\]

\[
\leq \mu(A \triangle (B \cap T^{-n}B)) + |\mu(B)^2 - \mu(A)|^2
\]

\[
\leq \mu(A \triangle B) + \mu(A \triangle T^{-n}B) + |\mu(B)^2 - \mu(A)|^2
\]

\[
= \mu(A \triangle B) + \mu(T^{-n}A \triangle T^{-n}B) + |\mu(B)^2 - \mu(A)|^2
\]

\[
\leq 2\varepsilon + |\mu(B)^2 - \mu(A)|^2 \leq 2\varepsilon + 2|\mu(A) - \mu(B)| \leq 4\varepsilon.
\]

This, being true for every \(\varepsilon > 0\), implies \(\mu(A) = \mu(A)^2\), hence \(\mu(A) \in \{0, 1\}\).

**Proposition 3.18** (Characterization of ergodicity). For a measure-preserving system \((X, \mu, T)\) the following assertions are equivalent.

(i) The system is ergodic.

(ii) Every \(A \subset X\) with \(\mu(A) > 0\) satisfies

\[
\bigcap_{k \in \mathbb{N}_0} \bigcup_{n \geq k} T^{-n}A = X \quad \text{up to a null set.}
\]

(iii) Every \(A \subset X\) with \(\mu(A) > 0\) satisfies

\[
\bigcup_{n \in \mathbb{N}_0} T^{-n}A = X \quad \text{up to a null set.}
\]

(iv) For each pair of sets \(A, B \subset X\) with \(\mu(A), \mu(B) > 0\) there is \(n \in \mathbb{N}\) with

\[
\mu(T^{-n}A \cap B) > 0.
\]

**Proof.** Suppose (i) and \(\mu(A) > 0\). For \(k \in \mathbb{N}_0\) define \(A_k := \bigcup_{n \geq k} T^{-n}A\). Then \(T^{-1}(A_k) = A_{k+1} \subset A_k\) and therefore \(A_k\) is an invariant set with \(\mu(A_k) \geq \mu(T^{-k}A) = \mu(A) > 0\). The assumption implies \(\mu(A_k) = 1\), and by intersecting \(\mu(\bigcap_{k \in \mathbb{N}} A_k) = 1\), i.e., (ii) follows. The implication (ii)⇒(iii) is trivial.
Assume (iii) and let \( B \subset X \) be measurable satisfying \( \mu(T^{-n}A \cap B) = 0 \) for each \( n \in \mathbb{N} \). For \( k = 0, 1 \) let again \( A_k := \bigcup_{n \geq k} T^{-n}A \). Since \( T^{-1}A_0 = A_1 \subset A_0 \), it follows that \( A_0 = A_1 \) up to a null set. We obtain

\[
0 = \mu \left( \bigcup_{n \in \mathbb{N}} (B \cap T^{-n}A) \right) = \mu(B \cap A_1) = \mu(B \cap A_0) = \mu(B \cap X) = \mu(B),
\]

proving (iv).

To see the implication (iv) \( \Rightarrow \) (i) take \( B := A^c \) in (iv) for an invariant set \( A \subset X \), to conclude that \( \mu(A) \in \{0, 1\} \).

**Remark 3.19.** Condition (iii) in the above characterization means that almost every \( x \in X \) visits every set \( A \) with positive measure at least once, and condition (ii) means that almost every \( x \in X \) visits such \( A \) infinitely often.

**Example 3.20** (Recurrence in random literature). We now discuss a concrete consequence of Proposition 3.18 which might appear somewhat counterintuitive at first glance. Imagine someone typing randomly on a typewriter which has 90 different typesetting symbols. The work starts now. This can be modeled by the one-sided Bernoulli shift \( B(1/90, \ldots, 1/90) \) which is ergodic by Proposition 3.17. An infinite piece of literature is then described by each of the sequences \( x \in \{0, \ldots, 89\}^\mathbb{N} \). Some of these sequences correspond to total nonsense, others contain poems of Goethe, romans of Joyce (or your favorite book), yet another contain some digits of \( \pi \), or a mixture of these, etc. Let \( a_1 a_2 \cdots a_N \) be the random text that can be seen in the background of the poster of this Internet Seminar. Since the cylinder set

\[
A := \{ x \in \{0, \ldots, 89\}^\mathbb{N} : x_1 = a_1, \ldots, x_N = a_N \}
\]

has positive measure (precisely \( 1/90^N \)), we see, by Proposition 3.18, that almost every \( x \in \{0, \ldots, 89\}^\mathbb{N} \) visits \( A \) infinitely many times under the shift-dynamics, meaning that the announcement of this Internet Seminar occurs in almost every infinite word infinitely often. This means that the person will almost surely type the poster infinitely often. (The poster was, however, produced by different methods.) What changes (and what does it mean for the typist) if we replace here the one-sided Bernoulli shift by a two-sided one?

### 3. Koopman operator and ergodicity

For a linear operator \( S : V \to V \) on a vector space \( V \) we defined the **fixed space** by

\[
\text{Fix}(S) := \{ f \in V : Sf = f \},
\]

and call each \( f \in \text{Fix}(S) \) **invariant under** \( S \). Note that for a measure-preserving system, constant functions are always invariant under the Koopman operator.

**Proposition 3.21** (Ergodicity via invariant functions). Let \( (X, \mu, T) \) be a measure-preserving system and let \( p \in [1, \infty) \). The Koopman operator on \( L^p(X, \mu) \) is also denoted by \( T \). The following assertions are equivalent.

(i) The system \( (X, \mu, T) \) is ergodic.

(ii) If \( f : X \to \mathbb{C} \) is measurable with \( f = f \circ T \mu\text{-a.e.} \), then there is \( c \in \mathbb{C} \) such that \( f = cf \mu\text{-a.e.} \).

(iii) Every invariant function \( f \in L^p(X, \mu) \) is constant.
(iv) Every invariant function \( f \in L^\infty(X, \mu) \) is constant.

**Proof.** (i)⇒(ii): Let \( f : X \to \mathbb{C} \) be measurable with \( Tf = f \) \( \mu \)-a.e., and assume without loss of generality that \( f \) is real-valued (otherwise we pass to the real and imaginary parts). For \( k \in \mathbb{Z} \) and \( n \in \mathbb{N}_0 \) define the measurable set
\[
A_{k,n} := \left[ \frac{k}{2^n} \leq f < \frac{k+1}{2^n} \right].
\]
This set is \( T \)-invariant, and thus either \( \mu(A_{k,n}) = 0 \), or \( \mu(A_{k,n}) = 1 \) by (i). Since for each fixed \( n \in \mathbb{N}_0 \) the sets \( A_{k,n} \) are pairwise disjoint for \( k \in \mathbb{Z} \) with \( \bigcup_{k \in \mathbb{Z}} A_{k,n} = X \), there is precisely one \( k(n) \in \mathbb{Z} \) with \( \mu(A_{k(n),n}) = 1 \). Then the set
\[
Z := \bigcap_{n \in \mathbb{N}_0} A_{k(n),n}
\]
has full measure, and for \( x, y \in Z \) we have \(|f(x) - f(y)| \leq 1/2^n \) for every \( n \in \mathbb{N} \), i.e., \( f(x) = f(y) \).

The implications (ii)⇒(iii)⇒(iv) are trivial. So it remains to prove (i) assuming (iv). Let \( A \subset X \) be invariant. We have \( T1_A = 1_{T-1_A} = 1_A \), i.e., the function \( 1_A \) is invariant under \( T \). By (iv), \( 1_A \) must be constant, so that \( \mu(A) \in \{0,1\} \).

The study of the ergodicity of torus rotations becomes now very easy.

**Proposition 3.22.** The torus rotation \((\mathbb{T}^d, m^d, \alpha) \) with \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{T}^d \) is ergodic if and only if \( \alpha_1, \ldots, \alpha_d \) are rationally independent, i.e., satisfy
\[
\left( (n_1, \ldots, n_d) \in \mathbb{Z}^d \quad \text{and} \quad \alpha_1^{n_1} \alpha_2^{n_2} \cdots \alpha_d^{n_d} = 1 \right) \quad \Rightarrow \quad n_1 = n_2 = \cdots = n_d = 0.
\]

**Proof.** The monomials \( z^n \) with \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \) given by \( z^n(z_1, \ldots, z_d) := z_1^{n_1} \cdots z_d^{n_d} \) form an orthonormal system in \( L^2(\mathbb{T}^d, m^d) \) and, by Proposition 1.9, even an orthonormal basis. Let \( f \) be an invariant function of the Koopman operator \( T \) on \( L^2(\mathbb{T}^d, m^d) \). Then expanding \( f \) with respect to the orthonormal basis leads to
\[
\sum_{n \in \mathbb{Z}^d} b_n z^n = f = \sum_{n \in \mathbb{Z}^d} a^n b_n z^n,
\]
as an \( L^2(\mathbb{T}^d, m^d) \)-convergent series, where \( a^n := \alpha_1^{n_1} \cdots \alpha_d^{n_d} \). By comparing the coefficients we obtain \( b_n = a^n b_n \) for every \( n \in \mathbb{Z}^d \).

If \( f \notin \mathcal{C}1 = \mathbb{C} z^0 \), then there is \( n \in \mathbb{Z}^d \setminus \{0\} \) with \( b_n \neq 0 \), implying \( 1 = a^n = \alpha_1^{n_1} \cdots \alpha_d^{n_d} \). This proves that in case of rational independence, \( \text{Fix}(T) = \mathcal{C}1 \), and ergodicity follows by Proposition 3.21.

For the converse implication, assume that \( \alpha_1, \ldots, \alpha_d \) satisfy \( \alpha_1^{n_1} \cdots \alpha_d^{n_d} = 1 \) with \( n = (n_1, \ldots, n_d) \neq (0, \ldots, 0) \). Then the corresponding monomial \( z^n \) is not constant and satisfies \( Tz^n = z^n \), implying that the system \((X, \mu, T)\) is not ergodic by Proposition 3.21.

The following gives information on the spectrum of the Koopman operator of an ergodic system. Recall for a vector space \( V \) and a linear operator \( S : V \to V \) the notation
\[
\sigma(S) := \{ \lambda \in \mathbb{C} : \exists f \in V \setminus \{0\} \text{ such that } Sf = \lambda f \}
\]
for the set of eigenvalues of \( S \), called the **point spectrum** of \( S \).
**Proposition 3.23** (Point spectrum of the Koopman operator for ergodic systems). Let \((X, \mu, T)\) be an ergodic measure-preserving system, and let \(T\) denote the Koopman operator on \(L^0(X, \mu)\). Then the set \(P\sigma(T) \cap \mathbb{T}\) of unimodular eigenvalues is a subgroup of \(\mathbb{T}\), each such eigenvalue has one-dimensional eigenspace, and a corresponding eigenfunction can be taken unimodular.

Since constant functions are always \(T\)-invariant, \(1 \in P\sigma(T)\) and, in particular, \(P\sigma(T) \neq \emptyset\).

**Proof.** Let \(\lambda \in P\sigma(T) \cap \mathbb{T}\) with corresponding eigenfunction \(f\). Since \(|f \circ T| = |f| \circ T\), we have that \(|f| \in \text{Fix}(T)\), so that, by ergodicity and by Proposition 3.21, \(|f|\) is a non-zero constant. Hence after scaling \(f\) can be taken to be unimodular.

Let \(\mu \in P\sigma(T) \cap \mathbb{T}\) have an eigenfunction \(g\) which, together with \(f\), is assumed to be unimodular. By the algebraic properties of the Koopman operator (see Proposition 2.22) we obtain that \(T(fg) = (f)T(g) = (\lambda f)g\). Since \(fg\) is unimodular and hence non-zero, it is an eigenfunction to the eigenvalue \(\lambda g\). If \(\mu = \lambda\), ergodicity yields \(fg = 1\), i.e., \(f = g\), showing the one-dimensionality of the eigenspace. Moreover, we see that for \(\lambda, \mu \in P\sigma(T) \cap \mathbb{T}\) also \(\mu\) and \(\lambda\) belong to \(P\sigma(T) \cap \mathbb{T}\), i.e., this set is subgroup of \(\mathbb{T}\). \(\blacksquare\)

4. More on invariant measures for topological systems

We now study the relation between topological and measure-preserving systems from the point of view of ergodicity properties. For this purpose we introduce the notion of absolute continuity with two central results, namely Lebesgue’s decomposition and the Radon–Nikodym theorem. For details we refer to Rudin’s book [RCA, 6.7–6.14].

Let \((X, \mathcal{X})\) be a measurable space, let \(\mu \in \mathfrak{M}(X, \mathcal{X})\) be a positive measure, and \(f \in L^1(X, \mathcal{X}, \mu) =: L^1(\mu)\). Then the measure \(\nu\) defined for \(A \in \mathcal{X}\) by

\[
\nu(A) := \int_A f \, d\mu
\]

satisfies \(\nu(A) = 0\) whenever \(\mu(A) = 0\). We use the notation \(f \cdot \mu\) for this measure \(\nu\).

The proof of the following lemma is a routine exercise, see, e.g., [RCA, Thm. 1.29]

**Lemma 3.24.** If \(\mu \in \mathfrak{M}(X, \mathcal{X})\) is a positive measure and \(\nu = f \cdot \mu\) for some \(f \in L^1(\mu)\), then for every bounded (or positive) measurable function \(g : X \to \mathbb{C}\)

\[
\int_X g \, d\nu = \int_X gf \, d\mu.
\]

**Definition 3.25.** (a) Let \(\mu, \nu \in \mathfrak{M}(X, \mathcal{X})\) be two positive measures. If each \(\mu\)-null set is at the same time a \(\nu\)-null set, then \(\nu\) is said to be absolutely continuous with respect to \(\mu\). We denote this by writing \(\nu \ll \mu\). If additionally \(\mu \ll \nu\), then we call the measures \(\mu\) and \(\nu\) equivalent, denoted by \(\mu \simeq \nu\).

(b) Let \(\mu, \nu \in \mathfrak{M}(X, \mathcal{X})\) be two complex measures. We say that \(\nu\) is absolutely continuous with respect to \(\mu\) if \(|\nu| \ll |\mu|\).

(c) A pair of complex measures \(\mu, \nu \in \mathfrak{M}(X, \mathcal{X})\) is called (mutually) singular if there is a set \(A \in \mathcal{X}\) with \(|\mu|(A) = |\nu|(A^c) = 0\). We denote this relation by writing \(\mu \perp \nu\).
We see that the measure $f \cdot \mu$ from the above is absolutely continuous with respect to $\mu$.

**Theorem 3.26** (Lebesgue decomposition & Radon–Nikodym theorem). Let $\nu, \mu \in \mathcal{M}(X, \mathcal{X})$ and let $\mu$ be positive.

(a) There are unique measures $\nu_\alpha$ and $\nu_\beta$ with $\nu_\beta \perp \mu$, $\nu_\alpha \ll \mu$ and $\nu = \nu_\alpha + \nu_\beta$.

(b) There is $f \in L^1(\mu)$ such that $\nu_\alpha = f \cdot \mu$, i.e.

$$\nu_\alpha(A) = \int_A f \, d\mu \quad \text{for each } A \in \mathcal{X}.$$ 

Moreover, if $\nu$ is positive, then so are $\nu_\alpha$, $\nu_\beta$ and $f$.

If $\nu \ll \mu$, then $\nu_\beta = 0$ in part (a), so that $\nu = f \cdot \mu$. The function $f$ is called the **Radon–Nikodym derivative** of $\nu$ with respect to $\mu$.

We now come return to topological systems and take closer look invariant measure. Let $(K, \mathcal{B})$ be a topological system. Let $M_1(K, \mathcal{B})$ be the set of $\mathcal{B}$-invariant, regular, Borel, probability measures. By the Krylov–Bogoljubov theorem (see Lecture 2) $M_1(K, \mathcal{B})$ is non-empty. An invariant, probability measure $\mu \in M_1(K, \mathcal{B})$ is called **ergodic** if $(K, \mathcal{B}(K), \mu, T)$ is an ergodic measure-preserving system.

**Proposition 3.27** (Ergodicity of invariant measures). Let $(K, \mathcal{B})$ be a topological system and let $\mu, \nu \in M_1(K, \mathcal{B})$.

(a) If $\mu$ is ergodic and $\nu \ll \mu$, then $\nu = \mu$.

(b) If $\mu \neq \nu$ and both are ergodic, then they are mutually singular.

(c) The measure $\mu \in M_1(K, \mathcal{B})$ is ergodic if and only if $\mu$ is an extreme point of the (weak* compact, convex) set $M_1(K, \mathcal{B})$.

**Proof.** (a) Since $\nu \ll \mu$ there is $f \in L^1(\mu)$ with $f \geq 0$ and $\nu = f \cdot \mu$ (Theorem 3.26).

We prove that $f = 1$ in $L^1(\mu)$. We show first that $A := \{f < 1\}$ is an invariant set for $(K, \mu, T)$. To see this we compute

$$\int_{A \setminus T^{-1}A} f \, d\mu + \int_{A \setminus T^{-1}A} f \, d\mu = \nu(A) = \nu(T^{-1}A) = \int_{A \setminus T^{-1}A} f \, d\mu + \int_{T^{-1}A \setminus A} f \, d\mu.$$ 

As a consequence we obtain

$$\int_{A \setminus T^{-1}A} f \, d\mu = \int_{T^{-1}A \setminus A} f \, d\mu.$$ 

Since $A \setminus T^{-1}A \subset \{f < 1\}$, $T^{-1}A \setminus A \subset \{f \geq 1\}$ and

$$\mu(A \setminus T^{-1}A) = \mu(A) - \mu(A \cap T^{-1}A) = \mu(T^{-1}A) - \mu(A \cap T^{-1}A) = \mu(T^{-1}A \setminus A),$$

we conclude from (3.3) that $\mu(A \setminus T^{-1}A) = \mu(T^{-1}A \setminus A)$, implying $A = T^{-1}A$ up to a $\mu$-null set, and the invariance of $A$ is proven.

By ergodicity of $\mu$ we have $\mu(A) \in \{0, 1\}$. If $\mu(A) = 1$ were true, then the inequality $\nu(X) = \int_X f \, d\mu = \int_A f \, d\mu < \mu(A) = 1$ would provide a contradiction. Thus $\mu(A) = 0$ must hold, meaning that $f \geq 1$ $\mu$-almost everywhere. Since $\nu = f \cdot \mu$ is a probability measure, we conclude $f = 1$ $\mu$-almost everywhere.

(b) By the Lebesgue decomposition $\nu = \nu_\alpha + \nu_\beta$ holds with uniquely determined positive measures $\nu_\alpha, \nu_\beta$ such that $\nu_\alpha \ll \mu$ and $\nu_\beta \perp \mu$. Since $\nu = T_\ast \nu = T_\ast \nu_\alpha + \ldots$
By Remark 3.28 there is an ergodic decom-
position implies $T_*\nu_\alpha = \nu$. Therefore $\nu_\alpha$ is an invariant measure with $\nu_\alpha \ll \mu$. If $\nu_\alpha = 0$, then $\nu = \nu_\alpha \ll \mu$, and we are done. If $\nu_\alpha \neq 0$, the probability measure $\nu_\alpha ||\nu_\alpha||$ is absolutely continuous with respect to $\mu$, so by part (a) $\nu_\alpha ||\nu_\alpha|| = \mu$. This implies $\mu \ll \nu_\alpha \leq \nu$. Since also $\nu$ is ergodic, this implies again by part (a) $\mu = \nu$.

(c) That $M_1(K,T)$ is convex and weak* compact is left as Exercise 3.9. Let $\mu \in M_1(K,T)$ and let $A$ be an invariant set of $(K,\mu,T)$. Then also $A^c$ is $T$-invariant. If $\mu(A) \in (0,1)$, then both $\mu_A/\mu(A)$ and $\mu_A/\mu(A^c)$ are $T$-invariant, probability measures, and $\mu$ is a non-trivial convex combination of these, showing $\mu \notin \text{Ex}(M_1(K,T))$. Conversely, suppose that $(K,\mu,T)$ is ergodic and $\mu = (1-t)\mu_1 + t\mu_2$ for some $t \in (0,1)$ and $\mu_1, \mu_2 \in M_1(K,T)$. Then $\mu_1 \ll \mu$, so from part (a) we deduce $\mu_1 = \mu$, and hence $\mu_2 = \mu$. 

**Remark 3.28.** By the Krein–Milman theorem, see Lecture 1, $\text{Ex}(M_1(K,T))$ is non-empty, i.e., there is always an ergodic measure $\mu \in M_1(K,T)$. Furthermore, we see that $M_1(K,T)$ is a singleton if and only if there is a unique ergodic measure $\mu \in M_1(K,T)$. In the latter case we call the topological system $(K,T)$ **uniquely ergodic**.

Next we connect invariant measures and invariant sets for topological systems.

**Proposition 3.29** (Invariant measures versus invariant sets). Let $(K,T)$ be a topological system.

(a) For $\mu \in M_1(K,T)$ we have $T\text{supp}(\mu) = \text{supp}(\mu)$. In particular, $\text{supp}(\mu)$ is a closed, invariant set in the topological system $(K,T)$.

(b) Let $L \subset K$ be a closed, invariant set. Then there is an ergodic measure $\mu \in M_1(K,T)$ with $\text{supp}(\mu) \subset L$.

(c) The system $(K,T)$ is minimal if and only if $\text{supp}(\mu) = K$ for every ergodic $\mu \in M_1(K,T)$.

**Proof.** (a) We first prove that $T\text{supp}(\mu) \subset \text{supp}(\mu)$. Let $y \in T\text{supp}(\mu)$ and let $U$ be an open set with $y \in U$. Then there is $x \in \text{supp}(\mu)$ with $Tx = y$, and by continuity there is an open set $V$ with $x \in V$ and $TV \subset U$. Since $V \subset T^{-1}(TV) \subset T^{-1}U$ and by the definition of the support, we obtain $0 < \mu(V) \leq \mu(T^{-1}U) = \mu(U)$. It follows that $y \in \text{supp}(\mu)$.

To see the converse inclusion $\text{supp}(\mu) \subset T\text{supp}(\mu)$ let $f \in C(K)$ vanish on the compact set $T\text{supp}(\mu)$ but otherwise arbitrary. Then

$$\int_K f \, d\mu = \int_K f \, dT_*\mu = \int_K f \circ T \, d\mu = \int_{\text{supp}(\mu)} f \circ T \, d\mu = 0.$$  

By Proposition 1.12 we obtain that $\text{supp}(\mu) \subset T\text{supp}(\mu)$.

(b) By Remark 3.28 there is an ergodic $\nu \in M_1(L,T)$. For $B \in B(K)$ define $\mu(B) := \nu(B \cap L)$. Then $\mu \in M_1(K,T)$ (why?) and $\text{supp}(\mu) \subset L$. To show that $\mu$ is ergodic, let $A \subset K$ be an invariant set for $(K,\mu,T)$. Then $0 = \mu(A \setminus T^{-1}A) = \nu((A \setminus L) \setminus (T^{-1}(A \cap L)))$. The ergodicity of $\nu$ implies $\mu(A) = \nu(A \cap L) \in \{0,1\}$.

(c) To show that under the asserted condition $(K,T)$ is minimal we can apply part (b). Conversely, suppose that $(K,T)$ is minimal. Then by part (a) $\text{supp}(\mu) = K$ must hold for every invariant, in particular, for every ergodic measure $\mu$. 


We finish this lecture with a higher-dimensional analogue of Kronecker’s theorem. We present a proof motivated by what has been said in Lecture 1.

**Proposition 3.30.** For \( a \in \mathbb{T}^d \) consider the torus rotation \((\mathbb{T}^d, a)\), and denote its Koopman operator on \( C(\mathbb{T}^d) \) by \( L_a \). Then for each \( f \in C(\mathbb{T}^d) \) the limit
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} L_a^n f
\]
exists in \( C(\mathbb{T}^d) \). If \( a_1, \ldots, a_d \) are rationally independent, then the limit equals \( \int_{\mathbb{T}^d} f \, dm^d \cdot 1 \).

**Proof.** For \( N \in \mathbb{N} \) let \( A_N := \frac{1}{N} \sum_{n=0}^{N-1} L_a^n \). Then \( A_N \in \mathcal{L}(C(\mathbb{T}^d)) \) with \( \|A_N\| \leq 1 \) for each \( N \in \mathbb{N} \). Therefore, it suffices to prove the convergence of \( A_N f \) for \( f \) in the dense subspace \( P(\mathbb{T}^d) \) of trigonometric polynomials (cf. Exercise 1.3). We may also restrict our attention to trigonometric monomials \( z^m \), where \( m \in \mathbb{Z}^d \). We then have
\[
L_a z^m = a_1^{m_1} \cdots a_d^{m_d} \cdot z^m.
\]
If \( b := a_1^{m_1} \cdots a_d^{m_d} = 1 \), we readily see that \( A_N z^m = z^m \) for each \( N \in \mathbb{N} \). Otherwise we obtain
\[
A_N z^m = \frac{1}{N} \sum_{n=0}^{N-1} b^n z^m = \frac{1}{N} \frac{b^N - 1}{b - 1} z^m \to 0 \quad \text{as} \quad N \to \infty.
\]
The last assertion follows by linearity and by the above calculation, since in the case of rational independence for every \( m \in \mathbb{Z}^d \setminus \{0\} \) the limit of \( A_N z^m \) equals \( 0 = \int_{\mathbb{T}^d} z^m \, dm^d \cdot 1 \), and the limit of \( A_N z^0 = 1 \) the limit equals \( 1 \).

As a corollary we obtain the promised result.

**Theorem 3.31** (Kronecker). The torus rotation \((\mathbb{T}^d, a)\) with \( a = (a_1, \ldots, a_d) \in \mathbb{T}^d \) is minimal if and only if \( a_1, \ldots, a_d \) are rationally independent. In this case \((\mathbb{T}^d, a)\) is uniquely ergodic, and the Haar measure is the unique (ergodic) invariant measure.

**Proof.** If \( a_1^{m_1} \cdots a_d^{m_d} = 1 \) for some \( m \in \mathbb{Z}^d \setminus \{0\} \), then
\[
F := \{ z \in \mathbb{T}^d : z_1^{m_1} \cdots z_d^{m_d} = 1 \}
\]
is a non-trivial, closed, invariant set. Therefore \((\mathbb{T}^d, a)\) is not minimal.

Conversely, suppose \( a_1^{m_1} \cdots a_d^{m_d} \neq 1 \) for each \( m \neq 0 \). Proposition 3.22 and the hypothesis yield that \( m^d \) is an ergodic measure for \((\mathbb{T}^d, a)\). Let \( \mu \in M_1(\mathbb{T}^d, a) \) be any ergodic measure. By the previous proposition and by the dominated convergence theorem \( \langle f, \mu \rangle = \langle A_N f, \mu \rangle \to \langle f, m^d \rangle (1, \mu) = \langle f, m^d \rangle \) as \( N \to \infty \) for every \( f \in C(\mathbb{T}^d) \). This implies \( \mu = m^d \). Since \( \text{supp}(m^d) = \mathbb{T}^d \), Proposition 3.29 yields that \((\mathbb{T}^d, a)\) is minimal.

The following characterization can be proven analogously to the handled special case of \( \mathbb{T}^d \) using more theory of compact, abelian groups, see, e.g., [EFHN, Thm. 10.13 and Prop. 14.21].

**Theorem 3.32** (Ergodicity and minimality for rotations). Let \((G, a)\) be a rotation system. Then the following assertions are equivalent.

(i) \((G, a)\) is minimal.
(ii) \((G, m_G, a)\) is ergodic.
(iii) \( \{a^n : n \in \mathbb{N}_0 \} \) is dense in \( G \).

In this case, \( G \) is abelian, \((G, a)\) is uniquely ergodic, and \( m_G \) is the unique invariant probability measure.
Exercises

Exercise 3.1 (Invariant sets). Let \((K, T)\) be an invertible topological system and let \(A \subset K\) be a non-empty, closed, invariant set. Show that there exists a non-empty, closed, two-sided invariant set \(B \subset A\).

Exercise 3.2 (Minimality). Prove the characterization of minimality via the orbits as explained in Remark 3.1.

Exercise 3.3 (Group rotations). Prove that if a group rotation \((G, a)\) is minimal, then \(G\) is abelian. Is the converse true?

Exercise 3.4. Prove Propositions 3.7 and 3.8.

Exercise 3.5 (Periodic points). Let \((K, T)\) be a topological system. A point \(x \in K\) is called periodic in \((K, T)\) if there is \(p \in \mathbb{N}\) such that \(T^p x = x\). Show that a periodic point is almost periodic. Is the converse true?

Exercise 3.6. Consider the compact space \(K = \{0, 1\}^\mathbb{N}\) and the one-sided, shift system \((K, T)\) from Example 2.24.

(a) Prove that a point \(x \in K\) is almost periodic if and only if for each finite subword \(y\) of \(x\) there is \(\ell \in \mathbb{N}\) such that the gap between each two subsequent occurrences of \(y\) is of length at most \(\ell\).

(b) Give an example of an almost periodic but not periodic point \(x \in K\) (cf. Exercise 3.5).

(c) Give an example of a point \(x \in K\) which is recurrent but not almost periodic (cf. Proposition 3.7).

Exercise 3.7 (Invariant sets). Let \((X, \mu, T)\) be a measure-preserving system. Prove that a set \(A \subset X\) is invariant if and only if \(A \subset T^{-1}A\) up to a null set, and if and only if \(T^{-1}A \subset A\) up to a null set. Prove that the \(T\)-invariant sets form a \(\sigma\)-algebra.

Exercise 3.8 (Ergodicity). Prove that a measure-preserving system \((X, \mu, T)\) is ergodic if and only if each measurable set \(A \subset X\) with \(T^{-1}A = A\) satisfies \(\mu(A) \in \{0, 1\}\).

Exercise 3.9 (Invariant measures). Let \((K, T)\) be a topological system. Show that \(M_1(K, T)\) of all \(T\)-invariant regular, Borel, probability measures on \(K\) is a convex and weak* compact subset of \(M(K)\).

Exercise 3.10 (Rotations on the unit disc). Consider the closed unit disc \(\overline{D} := \{z \in \mathbb{C} : |z| \leq 1\}\), and let \(a \in \mathbb{T}\) be not a root of unity. The rotation \(\tau_a : \overline{D} \to \overline{D}\), \(z \mapsto az\) is continuous, and gives rise to the invertible topological system \((\overline{D}, \tau_a)\). Describe all closed, \(\tau_a\)-invariant sets, all invariant probability measures, and determine the ergodic ones. Can you give a relation between Proposition 1.14(c) and Proposition 3.27(c) in this situation?
References


